

The correlation functions of power as a new proposition to describe power states in circuits with periodical voltage and current waveforms

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Abstract— The analysis of energy transfer in the circuits is based on the instantaneous power $p(t)$, which is the product of the two waveforms $u(t), i(t)$. An interesting thing is the search for mutual relations between these two waveforms. In the paper the authors made an attempt to describe power states in electrical circuits with periodic waveforms with the help of the correlation of the two functions. In mathematics correlation is the measure of similarity or mutual dependence between the two functions. Electrical energy is a simultaneous and mutual cooperation of voltage and current waveforms - therefore, the authors defined and examined the correlation function $\Psi(t)$ of these waveforms. On the basis of the defined correlation function between the voltage and the current, the authors introduced two new powers: the sine power function $\Psi_s(t)$ and the cosine power function $\Psi_c(t)$.

I. Introduction

In the paper it was explicitly demonstrated [3] that the power theories making use of Fourier's decomposition of voltages and currents can be applied only in linear circuits. Such decomposition results straight from the principle of superposition. It was also pointed out that the power theories based on the orthogonal decomposition of Fryze's voltage or current (and all derivative decompositions) have numerous defects and limitations.

The analysis of energy transfer in electrical circuits is based on the instantaneous power $p(t)$, which is the product of two waveforms: the voltage waveform $u(t)$ and the current waveform $i(t)$. The authors made an attempt to look for a mutual relation between these two waveforms. As a result of the analyses, new descriptions of energy states in electrical circuits with the help of the correlation of two functions were proposed.

II. A new proposal to describe energy states in linear circuits with any voltage and current waveforms.

The non-sinusoidal waveforms of voltage and current **in any given linear circuit** have such a property that the harmonics order of a Fourier series, which is the approximation of the non-sinusoidal voltage $u(t)$ is the same as the harmonics order of a Fourier series approximating the non-sinusoidal waveform of the current $i(t)$.

Generally, it can be stated that the harmonics order of voltage and current in a given circuit defines the properties of such a circuit. So then,

- if in the analyzed circuit the harmonics order of the voltage (n) is equal to the harmonics order of the current (m), that is $n = m$, then such a circuit can be a **linear or non-linear circuit**,
- if in the analyzed circuit the harmonics order of the voltage (n) is different from the harmonics order of the current (m), that is $n \neq m$, then such a circuit is certainly a **non-linear circuit**.

For the non-sinusoidal linear circuit the instantaneous voltage and current waveforms have the following shape

$$u(t) = \sum_n \sqrt{2} U_n \cdot \cos(n\omega t + \alpha_n) = \sum_n u_n(t) \quad (1)$$

$$i(t) = \sum_n \sqrt{2} I_n \cdot \cos(n\omega t + \beta_n) = \sum_n i_n(t) \quad (2)$$

Introducing the obligatory definitions of the RMS values of the non-sinusoidal waveforms (equations 1 and 2), we can write:

$$U = \sqrt{\frac{1}{T} \int_{\tau}^{\tau+T} u^2 dt} = \sqrt{\sum_n U_n^2} \text{ is the RMS value of the}$$

non-sinusoidal voltage (equation 1),

$$I = \sqrt{\frac{1}{T} \int_{\tau}^{\tau+T} i^2 dt} = \sqrt{\sum_n I_n^2} \text{ is the RMS value of the non-}$$

sinusoidal current (equation 2),

$S = U \cdot I$ is the apparent power equal to the maximal value of the active power P , which can be dissipated in the linear circuit at the same RMS values of the voltage U and the current I .

For the any circuit the instantaneous power $p(t)$ is defined as $p(t)=u(t)i(t)$. After using the equations (2) and (3), we obtain

$$p(t) = u(t) \cdot i(t) = [\sum_n \sqrt{2} U_n \cdot \cos(n\omega t + \alpha_n)] \cdot [\sum_n \sqrt{2} I_n \cdot \cos(n\omega t + \beta_n)] \quad (3)$$

Decomposing the current harmonics of the – in accordance with Shepherd – Zakikhani's decomposition – into two orthogonal components with regard to the voltage harmonics, the non-sinusoidal current waveforms $i(t)$ can be written down in the following form

$$i(t) = i_r(t) + i_s(t) = [\sum_n \sqrt{2} I_n \cdot \cos(\theta_n) \cdot \cos(n\omega t + \alpha_n)] + [\sum_n \sqrt{2} I_n \cdot \sin(\theta_n) \cdot \sin(n\omega t + \alpha_n)] \quad (4)$$

where:

$$i_r(t) = \sum_n \sqrt{2} I_n \cdot \cos(\theta_n) \cdot \cos(n\omega t + \alpha_n) \quad (5)$$

is the resistance component of the current $i(t)$,

$$i_{rs}(t) = \sum_n \sqrt{2} I_n \cdot \sin(\theta_n) \cdot \sin(n\omega t + \alpha_n) \quad (6)$$

is the reactive component of the current $i(t)$,

$$\theta_n = \alpha_n - \beta_n \quad (7)$$

is the shift angle of the initial phases of the “n”-th voltage and current harmonics.

Introducing the following designations of the effective values of the “n”-th harmonics of the active and reactive components of the current:

$$I_{pn} = I_n \cdot \cos(\theta_n) \quad (8)$$

$$I_{qn} = I_n \cdot \sin(\theta_n) \quad (9)$$

we can write
$$I_n^2 = I_{pn}^2 + I_{qn}^2 \quad (10)$$

which implies that

$$I_R^2 = \sum_n I_{pn}^2 \quad I_{rs}^2 = \sum_n I_{qn}^2 \quad I^2 = \sum_n I_n^2$$

and
$$I^2 = I_R^2 + I_{rs}^2 \quad (11)$$

If we multiply both the sides of the equation (10) by U_n^2 (and not as in Shepherd-Zakikhani by U^2), we obtain:

$$U_n^2 \cdot I_n^2 = U_n^2 \cdot I_{pn}^2 + U_n^2 \cdot I_{qn}^2 \quad (12)$$

Designating with: $U_n^2 \cdot I_n^2 = S_n^2$, $U_n^2 \cdot I_{pn}^2 = P_n^2$,

$$U_n^2 \cdot I_{qn}^2 = Q_n^2$$

we obtain the equation
$$S_n^2 = P_n^2 + Q_n^2 \quad (13)$$

The equation (13) is the result of the application of the principle of superposition for the “n”-th voltage and current harmonic. Summing with the sides of the equation (13), we obtain for all “n”

$$\sum_n S_n^2 = \sum_n (P_n^2 + Q_n^2) \quad (14)$$

or in the form that is called the *correlative equation of power*

$$S_s^2 = P_s^2 + Q_s^2 \quad (15)$$

The authors suggest naming the particular constituents of the equation (15) in the following way

$$S_s = \sqrt{\sum_n S_n^2} \quad \text{the correlative apparent power,} \quad (16)$$

$$P_s = \sqrt{\sum_n P_n^2} \quad \text{the correlative active power,} \quad (17)$$

$$Q_s = \sqrt{\sum_n Q_n^2} \quad \text{the correlative reactive power.} \quad (18)$$

The equation (15) is a conventional square equation of the power for the “n” harmonic in the linear circuits with the sinusoidal waveforms – therefore the authors suggest a letter “s” in the index – equations (16,17,18)

Calculating the apparent power S requires multiplying two sums of the squares of the RMS values of the voltage and current harmonics. In the linear circuit the order of these harmonics is the same, that is $n, m < N$. In order to follow the operation of multiplying, we assume that “n” is the order of the voltage harmonics, whereas “m” is the order of the current harmonics. With such assumptions we can write

$$S^2 = U^2 I^2 = \sum_n U_n^2 \sum_m I_m^2 = \sum_n U_n^2 I_n^2 + \sum_{n \neq m} U_n^2 I_m^2 = \sum_n S_n^2 + \sum_{n \neq m} U_n^2 I_m^2 \quad (19)$$

Introducing the notion of the *fixatory apparent power* HS_s (similar to [1,2,4,5])

$$HS_s = \sqrt{\sum_{m \neq n} U_m^2 I_n^2} \quad (20)$$

the equation (19) assumes the following form

$$S^2 = S_s^2 + HS_s^2 \quad (21)$$

The equation (21) is presented in the graphic form in Table 1

Table 1. The constituents of the equation (21)

X	I_1^2	I_2^2	...	I_n^2	...	I_{N-1}^2	I_N^2
U_1^2	$U_1^2 \cdot I_1^2$	$U_1^2 \cdot I_2^2$		$U_1^2 \cdot I_n^2$		$U_1^2 \cdot I_{N-1}^2$	$U_1^2 \cdot I_N^2$
U_2^2	$U_2^2 \cdot I_1^2$	$U_2^2 \cdot I_2^2$		$U_2^2 \cdot I_n^2$		$U_2^2 \cdot I_{N-1}^2$	$U_2^2 \cdot I_N^2$
\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots	\vdots
U_n^2	$U_n^2 \cdot I_1^2$	$U_n^2 \cdot I_2^2$		$U_n^2 \cdot I_n^2$		$U_n^2 \cdot I_{N-1}^2$	$U_n^2 \cdot I_N^2$
\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots	\vdots
U_{N-1}^2	$U_{N-1}^2 \cdot I_1^2$	$U_{N-1}^2 \cdot I_2^2$		$U_{N-1}^2 \cdot I_n^2$		$U_{N-1}^2 \cdot I_{N-1}^2$	$U_{N-1}^2 \cdot I_N^2$
U_N^2	$U_N^2 \cdot I_1^2$	$U_N^2 \cdot I_2^2$		$U_N^2 \cdot I_n^2$		$U_N^2 \cdot I_{N-1}^2$	$U_N^2 \cdot I_N^2$

Table 1 presents the results of multiplication of all the constituents of the equation (21), where $1 \leq n, m \leq N$. The sum of all the elements of Table 1 is equal to the value of S^2 , and the sum of the elements of the main diagonal (designated with the gray color) is equal to the value of

S_s^2 . The sum of the other elements of Table 1 is equal to the value of HS_s^2 .

Similarly, additional notions are introduced to the fixatory definition of the apparent power:

fixatory active power HP_s

$$HP_s^2 = \sum_{m \neq n}^N P_m P_n \quad (22)$$

and

fixatory reactive power HQ_s

$$HQ_s^2 = \sum_{m \neq n} Q_m Q_n \quad (23)$$

Similarly to the equation (21), we can write

Table 2. The constituents of the equations (24)

X	P_1	P_2	...	P_n	...	P_{N-1}	P_N
P_1	P_1^2	$P_1 P_2$		$P_1 P_n$		$P_1 P_{N-1}$	$P_1 P_N$
P_2	$P_2 P_1$	P_2^2		$P_2 P_n$		$P_2 P_{N-1}$	$P_2 P_N$
\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots	\vdots
P_n	$P_n P_1$	$P_n P_2$		P_n^2		$P_n P_{N-1}$	$P_n P_N$
\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots	\vdots
P_{N-1}	$P_{N-1} P_1$	$P_{N-1} P_2$		$P_{N-1} P_n$		P_{N-1}^2	$P_{N-1} P_N$
P_N	$P_N P_1$	$P_N P_2$		$P_N P_n$		$P_N P_{N-1}$	P_N^2

Table 2 includes the results of multiplication of all the elements $P_m \cdot P_n$. The sum of all the elements of Table 2 is equal to the value P^2 . Summing only by the main diagonal (the gray color) is equal to the value P_s^2 whereas the sum of all the other elements is equal to the value of HP_s^2 .

Similar considerations refer to the reactive powers of particular harmonics.

On the basis of the equations (13)(19)(20)(21)(24) and (25), we can write

$$S^2 = S_s^2 + HS_s^2 = P_s^2 + Q_s^2 + HS_s^2 \quad (26)$$

Transforming the equation (26), we obtain

$$S^2 = P^2 + Q_b^2 + (HS_s^2 - HQ_s^2 - HP_s^2) = P^2 + Q_b^2 + D^2 \quad (27)$$

where:

$D^2 = HS_s^2 - HQ_s^2 - HP_s^2$ is the distortion power of Budeanu.

III. The correlation function of the non-sinusoidal, periodical voltage and current waveforms in the linear circuit

In mathematics correlation is the measure of similarity or mutual dependence between the two functions $x(t), y(t)$. The correlation of the two functions is used for the integral in the form

$$(y \circ x)[t] \equiv \frac{1}{T} \int_0^T x(\tau) \cdot y(\tau - t) d\tau, \quad (28)$$

$$(x \circ y)[t] \equiv \frac{1}{T} \int_0^T x(\tau - t) \cdot y(\tau) d\tau$$

$$P^2 = P_s^2 + HP_s^2 \quad (24)$$

and

$$Q^2 = Q_b^2 = Q_s^2 + HQ_s^2 \quad (25)$$

where: Q_b is the power defined by Budeanu.

Table 2 presents the results of multiplication of all the constituents of the equation (24), where $1 \leq n, m \leq N$

The correlation has such a property that $(x \circ y)[t] = (y \circ x)[-t]$.

The analysis of energy transfer in linear circuits is based on the instantaneous power $p(t)$, which is the product of two waveforms. An interesting thing is the search for mutual relations between these two waveforms with the help of the correlation of two periodic time waveforms $u(t), i(t)$ of the period T .

Assuming that the time waveforms $u(t), i(t)$ have the forms presented by the equations (1 and 2), the correlation function, which is called by the authors *the power function* and is designated by the symbol $\Psi(t)$, is equal to

$$\Psi(+t) = (i \circ u)[t] = \sum_n I_n U_n \cdot \cos(n\omega t + \theta_n) \quad (29)$$

$$\Psi(-t) = (u \circ i)[t] = \sum_n I_n U_n \cdot \cos(n\omega t - \theta_n) \quad (30)$$

Supposing *the cosine power function* is defined in the following way

$$\Psi_c(t) \equiv \frac{1}{2} [\Psi(-t) + \Psi(+t)] \quad (31)$$

whereas *the sine power function* has the following form

$$\Psi_s(t) \equiv \frac{1}{2} [\Psi(-t) - \Psi(+t)] \quad (32)$$

Putting the equations (29) and (30) to the equations (31) and (32), we obtain

$$\Psi_c(t) = \sum_n [I_n U_n \cdot \cos(\theta_n)] \cdot \cos(n\omega t) \quad (33)$$

$$\Psi_s(t) = \sum_n [I_n U_n \cdot \sin(\theta_n)] \cdot \sin(n\omega t) \quad (34)$$

The power functions defined by the equations (29)(30)(31)(32) and (33)(34) have very interesting properties, so:

1° $\Psi(0) = \Psi_c(0) = \sum_n I_n U_n \cdot \cos(\theta_n) = P$ that is the power function for $t = 0$ is equal to the active power P

2° $\sqrt{2} \cdot \|\Psi(t)\| = \sqrt{2} \cdot \|\Psi_c(t) + \Psi_s(t)\| = S_s$

3° $\sqrt{2} \cdot \|\Psi_c(t)\| = P_s$

4° $\sqrt{2} \cdot \|\Psi_s(t)\| = Q_s$

5° $\frac{1}{\omega} \cdot \frac{d}{dt} \Psi_s(t) \Big|_{t=0} = -\frac{\omega}{2} \int_0^T \Psi_s(t) dt = Q_l$ is the reactive power defined by Illović

6° $Q_{B_n} = \frac{2}{T} \int_0^T \Psi_s(t) \cdot \sin(n\omega t) dt$ {it must be noticed

that this equation is identical with the formula for measuring the odd coefficients of a Fourier series of the function $\Psi_s(t)$ }. Hence:

$Q_B = \sum_{n=1}^N Q_{B_n}$ is the power defined by Budeanu.

7° $P_n = \frac{2}{T} \int_0^T \Psi_c(t) \cdot \cos(n\omega t) dt$ {it must be noticed that

this equation is identical with the formula for measuring the even coefficients of a Fourier series of the function $\Psi_c(t)$ }. Hence:

$$P = \sum_{n=0}^N P_n$$

8° $\sum_n I_n U_n \cdot \cos(\theta_n) \cdot [1 - \cos(2n\omega t)] = \Psi_c(0) - \Psi_c(2t)$

9° $\sum_n [I_n U_n \cdot \sin(\theta_n)] \cdot \sin(2n\omega t) = \Psi_s(2t)$

10° – Assuming the simplified notation $U' = \left\| \frac{d}{dt} u(t) \right\|$

and $\Psi'_s(0) = \frac{d}{dt} \Psi_s(t) \Big|_{t=0}$, it is possible to present the optimal Shepherd – Zakikhani's capacity of in the following form:

$$C_{opt} = \frac{\Psi'_s(0)}{\omega \cdot U'^2} \quad (35)$$

11° From the features 6°, 7° it results that it is possible to construct the complex power function $\underline{\Psi}(t)$:

$$\underline{\Psi}(t) = \Psi_c(t) + j\Psi_s(t) \quad (36)$$

The complex power function $\underline{\Psi}(t)$ has such an interesting property that the harmonics coefficients of the real part of this function $\{\psi_c(t)\}$ are the n-th active powers P_n , whereas, the harmonics coefficients of the imaginary $\{\psi_s(t)\}$ part are the n-th reactive powers of Budeanu Q_{B_n} .

IV. Summary

The analysis of energy transfer in the circuits is based on the instantaneous power $p(t)$, which is the product of the two waveforms $u(t), i(t)$. An interesting thing is the search for mutual relations between these two

waveforms. In the paper the authors made an attempt to describe power states in electrical circuits with periodic waveforms with the help of the correlation of the two functions. In mathematics correlation is the measure of similarity or mutual dependence between the two functions. Electrical energy is a simultaneous and mutual cooperation of voltage and current waveforms - therefore, the authors defined and examined the correlation function $\Psi(t)$ of these waveforms. On the basis of the defined correlation function between the voltage and the current, the authors introduced two new powers: the sine power function $\Psi_s(t)$ and the cosine power function $\Psi_c(t)$.

In the paper it was demonstrated that the power functions have very interesting properties. For linear circuits the coefficients of a Fourier series of the cosine power function $\Psi_c(t)$ are the active powers of the nth harmonics, which can be written down as $a_n = P_n = U_n I_n \cos(\varphi_n)$. On the other hand, the coefficients of a Fourier series of the sine power function $\Psi_s(t)$ are the reactive powers of the nth harmonics that were written down in the form $b_n = Q_n = U_n I_n \sin(\varphi_n)$. The authors presented also several other dependences for the sine and cosine power function.

In the authors' opinion the proposition to make use of the correlation function $\Psi(t)$ is a new, interesting way to describe power states of electrical circuits with the periodic waveforms of the voltage and the current of any shape.

V. References

- [1] Cakareski Z., Emanuel A.E. *On the physical meaning of nonactive powers in three-phase systems*. IEEE Power Engineering Review, July 1999
- [2] Emanuel A.E. *Poynting vector and the physical meaning of nonactive powers*. IEEE Transactions on Instrumentation and Measurement, vol.54,no.4, August 2005, pp1457-1462
- [3] Hartman M., Hashad M. *A few remarks about transfer of the electrical energy through periodical current and voltage waveforms*. Proceedings of ISNCC 2008
- [4] <http://documents.wolfram.com/applications/insydes/Appendix/Fixator.html>
- [5] <http://www.powerstandards.com/PQTeachingToy>. Index.htm